SOME CLASS OF CHI-SQUARE MIXTURE OF THE TRANSFORMED BETA FAMILY DISTRIBUTIONS

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Abstract. We introduce a class of chi-square mixture of the transformed Beta family distributions which are the Chi-square mixture distributions of Pareto, Loglogistic and Burr. The main results in this study are as follows: (i) the definition of the Chi-square mixture of Pareto, Loglogistic and Burr distributions, (ii) derivation of some properties of each distributions, that is, moments and the shape characteristics, also (iii) relationships of Chi-square mixture of Pareto distribution, Chi-square mixture of Loglogistic distribution and Chi-square mixture of Burr distributions were given.

1. Introduction

Mixture of distributions were studied by a number of authors since 1894. According to Blischke [1] a mixture of distribution is a weighted average of probability distribution with positive weights that sum to one. The distributions thus mixed are called components of mixture. The weights themselves incorporate a probability distribution called the mixing distribution. For this property of weights, a mixture is again a probability distribution. Pearson [10] was the pioneer in the field of mixture of distributions who reflected on the mixture of two normal distributions. After a long gap, some basic properties of mixture distribution were studied by Robin [12], Mendhandall and Haider [8]. Rider [11] published a paper on the method of moment applied to a mixture of two exponential distributions. Mixture of two geometric distributions was studied by Daniele [5]. Haselblad [7] studied in greater detail the finite mixture of distribution from the exponential family.

Beta mixture of binomial, gamma mixture of poisson and poisson mixture of binomial distribution were considered by David [6]. Roy et al. [13], [15] defined and studied poisson mixture of distribution and negative binomial mixture of distribution. Other authors [2, 3, 4], considers mixed distributions which they called Laplace mixture, Rayleigh mixture, F, and Dual mixture of distributions. In this study, we defined the Chi-square mixture of distributions of some class of the transformed beta family distribution, that is, Chi-square mixtures of Pareto, Loglogistic and Burr and studied some of their properties.

2. Preliminaries

We shall now present how distributions are being mixed. Roy et al [16], defined mixture distribution by taking \( \theta \), the parameter of a family of distributions, given by the density function \( f(x; \theta) \), is itself subject to the change variation.

\[ f(x; \theta) = \sum \lambda(k) f_k(x) \]

\( \lambda(k) \) is the weight or mixing density of the mixture, and the \( f_k(x) \) is the density of the th\( \omega \) component of the mixture, and the \( \lambda(k) \) is the weight or mixing density of the mixture, and the \( f_k(x) \) is the density of the th\( \omega \) component of the mixture.
For finite mixture, the general formula is

$$
\sum_{k=1}^{p} f(x; \theta_k)g(\theta_k).
$$

The infinite analogue is given by

$$
\int f(x; \theta)g(\theta)d\theta,
$$

where $g(\theta)$ is a density function.

Consider the following definition from [18].

**Definition 2.1.** A random variable $X$ is said to have Chi-square mixture distribution with $v$ degrees of freedom and parameters $\theta$ and $p$, if its density function is given by

$$
f_X(x; v, \theta, p) = \int_{0}^{\infty} e^{-\frac{x^2}{2v}} (\chi^2)^{\frac{v}{2} - 1} g(x; \theta, p) \chi^2; \quad 0 < x < \infty,
$$

where $g(x; \theta, p)$ is a probability density function. The name of Chi-square mixture distribution comes from the fact that the distribution in Equation (3) is the weighted average of $g(x; \theta, p)$ with weights equal to the ordinates of chi-square distribution.

3. Main Result

Here, we define the Chi-square mixture of Pareto, Loglogistic and Burr distribution and obtain some properties these distributions. We also prove that Chi-square mixture of Pareto distribution and Chi-square mixture of Loglogistic distribution were both special cases of Chi-square mixture of Burr distribution. Moreover, the Chi-square mixed of Pareto, Loglogistic and Burr distributions were general than that of their unmixed distribution. We now give the definition of the Chi-square mixture of Pareto.

**Definition 3.1.** If a random variable $X$ is said to have a Chi-square mixture of Pareto distribution if the density function is given by

$$
f_X(x; v, \alpha, \theta) = \int_{0}^{\infty} e^{-\frac{x^2}{2v}} \frac{\alpha}{\Gamma\left(\frac{v}{2}\right)} (\chi^2 + \theta)^{\alpha \chi^2 + \theta} x^{\alpha - 1} d\chi^2; \quad 0 < x < \infty,
$$

with $v$ degrees of freedom, and parameters $\alpha, \theta > 0$ such that $\int_{-\infty}^{\infty} f_X(x; v, \alpha, \theta) = 1$.

**Theorem 3.1.** If $X$ follows a Chi-square mixture of Pareto distribution with $v$ degrees of freedom and parameters $\alpha, \theta > 0$ then the $r$th raw moment about the origin is given by

$$
\mu'_r = \int_{0}^{\infty} e^{-\frac{x^2}{2v}} \frac{\alpha}{\Gamma\left(\frac{v}{2}\right)} (\chi^2 + \theta)^{\alpha \chi^2 + \theta} x^{\alpha - 1} d\chi^2.
$$

Hence, the mean is

$$
\mu'_1 = \frac{v + \theta}{(\alpha - 1)},
$$
and the variance is
\[ \sigma^2 = \frac{\alpha v + 4 + 2\theta - 4v + \alpha \theta^2}{(\alpha - 1)^2(\alpha - 2)}. \]

Furthermore, the Skewness, \( \beta_1 \), is
\[ \frac{\mu_3}{\mu_2^3} = \left[ \frac{6(\alpha - 1)^2(v + 4)}{(\alpha - 1)^3(\alpha - 2)(\alpha - 3)} \right]^2 \]
\[ \left[ \frac{(\alpha - 1)^2(\alpha - 2)}{\alpha(v + 4 + 2\theta) - 4v + \alpha \theta^2} \right]^3 \]
while the Kurtosis, \( \beta_2 \), is
\[ \frac{\mu_4}{\mu_2^2} = \left[ \frac{24}{(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)} \right] \left( \frac{v + 4}{(v + 2)(v + 4)} + 4\theta v + 2\theta^2 \right) \]
\[ + 6\theta^2 v + 4v + 2\theta^2 \]
\[ - \left( \frac{6(\alpha - 1)^2(v + 4)}{(\alpha - 1)^3(\alpha - 2)(\alpha - 3)} \right) \]
\[ - \left( \frac{6\theta^2 v + 4v + 2\theta^2}{(\alpha - 1)^3(\alpha - 2)(\alpha - 3)} \right) \]
\[ + 6 \left( \frac{(\alpha v + 4 + 2\theta) - 4v + \alpha \theta^2}{(\alpha - 1)^2(\alpha - 2)} \right) \left( \frac{v + \theta}{\alpha - 1} \right)^2 - 3 \left( \frac{v + \theta}{\alpha - 1} \right)^4 \]
\[ \left( \frac{(\alpha - 1)^2(\alpha - 2)}{(\alpha v + 4 + 2\theta) - 4v + \alpha \theta^2} \right)^2 \]

Proof. Let \( X \) be a random variable with density of Chi-square mixture of Pareto distribution. Then the \( r \)th raw moment, \( \mu'_r \), is given by
\[ E[X^r] = \int_0^\infty \int_0^\infty \frac{e^{-\frac{x^2}{2}}(\chi^2)^{\frac{\alpha r}{2}}}{2\pi^{\frac{r}{2}}} \frac{x^r}{(x + (\chi^2 + \theta)^{\alpha + 1}) \delta} dx d\chi. \]

Since we are working with double integral, we first consider the following expression
\[ \lim_{c \to \infty} \int_0^c \frac{x^r}{(x + (\chi^2 + \theta)^{\alpha + 1}) \delta} dx. \]
It can be shown that Equation (7) is equal to
\[ \frac{r!(\chi^2 + \theta)^{\alpha - 1}}{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - r)}. \]
Hence, Equation (6) is just
(8) \[ \mu'_r = \int_0^{\infty} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{r}{2} - 1} \frac{r!(\chi^2 + \theta)^r}{(\alpha - 1)(\alpha - 2) \cdots (\alpha - r)} d\chi^2. \]

Now, if \( r = 1 \), Equation (8) becomes

\[ \mu'_1 = \frac{v + \theta}{(\alpha - 1)}. \]

When \( r = 2 \), we have

\[ \mu'_2 = \frac{2v^2 + 4v + 4\theta v + 2\theta^2}{(\alpha - 1)(\alpha - 2)}. \]

Hence the variance, \( \sigma^2 \), is

\[ \sigma^2 = \frac{\alpha v(v + 4 + 2\theta) - 4v + a\theta^2}{(\alpha - 1)^2(\alpha - 2)}. \]

Similarly, if \( r = 3 \) we get

\[ \mu'_3 = \frac{6}{2^\frac{3}{2} \Gamma\left(\frac{3}{2}\right)(\alpha - 1)(\alpha - 2)(\alpha - 3)} \lim_{b \to \infty} \int_0^{b} e^{-\frac{v}{2} \left(2\pi\right)^{\frac{3}{2} - 1} (2\pi + \theta)^3 (2d\pi)}, \]

which is just equal to

\[ \mu'_3 = \frac{6(v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)}. \]

Now, substituting the values of \( \mu'_3, \mu'_2 \) and \( \mu'_1 \) to the third central moment, \( \mu_3 \), which is given by, \( \mu'_3 - 3\mu'_2 \mu'_1 + 2\mu'_1^3 \), we have

\[ \mu_3 = \frac{6(v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} \frac{\left(\frac{v + \theta}{(\alpha - 1)}\right)^3}{-3 \left(\frac{v + \theta}{(\alpha - 1)}\right) \left[\frac{2v^2 + 4v + 4\theta v + 2\theta^2}{(\alpha - 1)(\alpha - 2)}\right]^2 + \left[\frac{v + \theta}{(\alpha - 1)}\right]^3}. \]

Therefore,

\[ \mu_3 = \frac{6(\alpha - 1)^2 \left[v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3\right]}{(\alpha - 1)^3(\alpha - 2)(\alpha - 3)} \frac{3(v + \theta)(\alpha - 1)(\alpha - 3)(2v^2 + 4v + 4\theta v + 2\theta^2) + 2(v + \theta)^3(\alpha - 2)(\alpha - 3)}{(\alpha - 1)^3(\alpha - 2)(\alpha - 3)}. \]

For the skewness, \( \beta_1 \), which is given by \( \frac{\mu'_3}{\mu'_2} \), we have
Let \[ \text{Corollary 3.1.} \]

Now, the kurtosis, \( \beta \)

To obtain for kurtosis, we will need the fourth central moment, \( \mu_4 \), which is given by \( \mu_4' - 4\mu_1'\mu_3' + 6\mu_2'\mu_4' - 3\mu_4'' \), then

\[
\frac{24}{(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)} \left( v(v + 2)(v + 4)(v + 6) + 4\theta v(v + 2)(v + 4) \right) \\
+ 6\theta^2 v(v + 2) + 4\theta^3 v + \theta^4 \\
+ \frac{6(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)(\alpha - 4)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} \\
- \frac{3(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)(\alpha - 4)}{(\alpha - 1)^2(\alpha - 2)(\alpha - 3)} \\
+ \frac{6(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)}{(\alpha - 1)^2(\alpha - 2)(\alpha - 3)} \left( \frac{v + \theta}{\alpha - 1} \right)^2 \\
- 3 \left( \frac{v + \theta}{\alpha - 1} \right)^4.
\]

Now, the kurtosis, \( \beta_2 \), which is given by \( \frac{\mu_4'}{\mu_2^2} \) is

\[
\frac{24}{(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)} \left( v(v + 2)(v + 4)(v + 6) + 4\theta v(v + 2)(v + 4) \right) \\
+ 6\theta^2 v(v + 2) + 4\theta^3 v + \theta^4 \\
+ \frac{6(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)(\alpha - 4)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)} \\
- \frac{3(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)(\alpha - 4)}{(\alpha - 1)^2(\alpha - 2)(\alpha - 3)} \\
+ \frac{6(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)}{(\alpha - 1)^2(\alpha - 2)(\alpha - 3)} \left( \frac{v + \theta}{\alpha - 1} \right)^2 \\
- 3 \left( \frac{v + \theta}{\alpha - 1} \right)^4.
\]

\[ \Box \]

**Corollary 3.1.** Let \( v = 0 \). Then all the values of \( \mu_1' \), \( \sigma^2 \), \( \mu_2' \), \( \mu_3' \), \( \mu_4' \), \( \mu_3 \), \( \mu_4 \), \( \beta_1 \) and \( \beta_2 \) are true for Pareto distribution with parameters \( \alpha \) and \( \theta \).

**Remark 3.1.** The shape characteristics of the Chi-square mixture of Pareto distribution are determined by:
i. coefficient of skewness

\[
\gamma_1 = \frac{\mu^3}{\mu_2^{3/2}} = \frac{6(v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)}
\]

\[
- 3(v + \theta)(\alpha - 1)(\alpha - 3)(2v^2 + 4v + 4\theta v + 2\theta^2) + 2(v + \theta)^3(\alpha - 2)(\alpha - 3)
\]

\[
\left(\frac{(\alpha - 1)^3(\alpha - 2)^3}{\sqrt{\alpha v(v + 4 + 2\theta) - 4v + \alpha\theta^2}}\right)^{3(\alpha - 2)^3}
\]

ii. coefficient of kurtosis

\[
\gamma_2 = \beta_2 - 3
\]

\[
= \left(24(\alpha - 2)(\alpha - 1)^3\left[v + 2(v + 6) + 4\theta v(v + 2)(v + 4) + 6\theta^2 v(v + 2) + 4\theta^3 v + \theta^4\right]\right)
\]

\[
- \left[24(v + \theta)(\alpha - 1)^2(\alpha - 4)\left[v + 2(v + 4) + 3\theta v(v + 2) + 3\theta^3 v + \theta^3\right]\right]
\]

\[
+ 6(v + \theta)^2(\alpha - 1)(\alpha - 3)(\alpha - 4)(2v^2 + 4v + 4\theta v + 2\theta^2) - 3(v + \theta)^4(\alpha - 2)(\alpha - 3)(\alpha - 4)
\]

\[
\left(\frac{1}{(\alpha - 3)(\alpha - 4)(\alpha v(v + 4 + 2\theta) - 4v + \alpha\theta^2)^{\gamma}}\right)^3 - 3
\]

Since the value of \(\gamma_1\) and \(\gamma_2\) are always positive, then the density of Chi-square mixture of Pareto distribution is skewed to the right and the shape is leptokurtic.

Now, consider the following definition.

**Definition 3.2.** A random variable \(X\) is said to have a Chi-square mixture of Loglogistic distribution if the density function is given by

\[
f_X(x; v, \alpha, \gamma) = \int_{0}^{\infty} e^{-\frac{x^2}{2} \left(\frac{\chi^2}{2}\right)^{\gamma-1}} \frac{\gamma(\frac{x}{\alpha + x})^{\gamma}}{x[1 + \left(\frac{x}{\alpha + x}\right)^\gamma]^2} d\chi^2; \quad 0 < x < \infty,
\]

with \(v\) degrees of freedom and parameters \(\alpha, \gamma > 0\) such that \(\int_{-\infty}^{\infty} f_X(x; v, \alpha, \theta) = 1\).

**Theorem 3.2.** If \(X\) follows a Chi-square mixture of Loglogistic distribution with \(v\) degrees of freedom and parameters \(\alpha, \gamma > 0\), then the \(r\)th raw moment about the origin is given by

\[
\mu'_r = \int_{0}^{\infty} e^{-\frac{x^2}{2} \left(\frac{\chi^2}{2}\right)^{\gamma}} \frac{\gamma(\frac{x}{\alpha + x})^{\gamma}}{x[1 + \left(\frac{x}{\alpha + x}\right)^\gamma]^2} \Gamma(1 + \frac{r}{\gamma})\Gamma(1 - \frac{r}{\gamma})(\chi^2 + \theta)^r d\chi^2.
\]

Hence, the mean is

\[
\mu'_1 = (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}),
\]

and variance is
Furthermore, the Skewness, $\beta_1$, is given by

$$\frac{\mu_3}{\mu_2^2} = \left[ \frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})(\theta^2 + 2\theta v + v(v + 2)) - \Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})(\theta^2 + 2\theta v + v(v + 2))}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})(\theta^2 + 2\theta v + v(v + 2)) - \Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})^2} \right]^3,$$

and the Kurtosis, $\beta_2$, is given by

$$\frac{\mu_4}{\mu_2^2} = \left( \frac{\Gamma(1 + \frac{4}{\gamma})\Gamma(1 - \frac{4}{\gamma})}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})}\right)^\frac{3}{2} \frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})(\theta^2 + 2\theta v + v(v + 2)) - \Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})^2}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})(\theta^2 + 2\theta v + v(v + 2)) - \Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})^2} \right)^2.$$

**Proof.** Let $X$ be a random variable with density of Chi-square mixture of Loglogistic distribution. Then the $r$th raw moment about the origin is given by

$$\mu'_r = \int_0^\infty \int_0^\infty e^{-\frac{x^2}{2\gamma}(\chi^2 + 1)^{\frac{1}{\gamma}}} [\gamma(\frac{\chi^2}{\gamma} + x^2)]^{\gamma - 1} x^{r-1} dx d\chi^2,$$

which is equivalent to,

$$\mu'_r = \int_0^\infty e^{-\frac{x^2}{2\gamma}(\chi^2 + 1)^{\frac{1}{\gamma}}}[\gamma(1 + \frac{r}{\gamma})\Gamma(1 - \frac{r}{\gamma})(\chi^2 + \theta)^r d\chi^2.$$

If $r = 1$, then from the Equation (10), we get

$$\mu'_1 = \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})}{2\gamma} \left[ \theta \lim_{b \to \infty} \int_0^b e^{-u(2a)^{\frac{1}{\gamma}} d\chi^2} + \gamma \lim_{b \to \infty} \int_0^b e^{-u(2a)^{\frac{1}{\gamma}} d\chi^2} \right].$$
which is just,
\[ \mu' = (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}). \]

If \( r = 2 \), then Equation (10) becomes
\[ \mu_2' = \Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma}) \left[ v(v + 2) + 2\theta v + \theta^2 \right]. \]

Thus, the variance is
\[ \sigma^2 = \Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma}) \left[ v(v + 2) + 2\theta v + \theta^2 \right] - \left[ (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}) \right]^2. \]

If \( r = 3 \), Equation (10) becomes
\[ \mu_3' = \Gamma(1 + \frac{3}{\gamma})\Gamma(1 - \frac{3}{\gamma}) \left[ v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3 \right] \]

Similarly, if \( r = 4 \), we get
\[ \mu_4' = \Gamma(1 + \frac{4}{\gamma})\Gamma(1 - \frac{4}{\gamma}) \left[ v(v + 2)(v + 4)(v + 6) + 4\theta v(v + 2)(v + 4) + 6\theta^2 v(v + 2) + 4\theta^3 v + \theta^4 \right] \]

Therefore, we can compute \( \mu_3' \), which is given by \( \mu_3' = 3\mu_1'\mu_2' + 2\mu_1'^3 \), and that is equal to
\[ \left( \Gamma(1 + \frac{3}{\gamma})\Gamma(1 - \frac{3}{\gamma}) \left[ v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3 \right] \right) \]

\[ - 3 \left[ (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}) \left[ \Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})(\theta^2 + 2\theta v + v(v + 2)) \right] \right] \]

\[ + 2 \left[ (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}) \right]^3. \]

Similarly, for \( \mu_4' \), which is given by \( \mu_4' = 4\mu_1'\mu_3' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 \), we have
\[ \left( \Gamma(1 + \frac{4}{\gamma})\Gamma(1 - \frac{4}{\gamma}) \left[ v(v + 2)(v + 4)(v + 6) + 4\theta v(v + 2)(v + 4) + 6\theta^2 v(v + 2) + 4\theta^3 v + \theta^4 \right] \right) \]

\[ - 4 \left[ (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}) \right] \]

\[ \left( \Gamma(1 + \frac{3}{\gamma})\Gamma(1 - \frac{3}{\gamma}) \left[ v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3 \right] \right) \]

\[ + 6 \left( \Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma}) \left[ v(v + 2) + 2\theta v + \theta^2 \right] \right) \left[ (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}) \right]^2 \]

\[ - 3 \left[ (\theta + v)\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma}) \right]^4. \]

Hence, the skewness is
\[ \beta_1 = \frac{\mu_2^2}{\mu_2^3} \]
\[ = \left( \frac{1}{\Gamma(1 + \frac{3}{\gamma})\Gamma(1 - \frac{3}{\gamma})} \right) \frac{(v+2)(v+4) + 3\theta v(v+2) + 3\theta^2 v + \theta^3}{v(v+2)(v+4)} \]
\[ = \left( \frac{1}{\Gamma(1 + \frac{3}{\gamma})\Gamma(1 - \frac{3}{\gamma})} \right) \frac{(v+2)(v+4) + 3\theta v(v+2) + 3\theta^2 v + \theta^3}{v(v+2)(v+4)} \]
\[ - 3 \left( \frac{1}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})} \right) \frac{(v+2)(v+4) + 3\theta v(v+2) + 3\theta^2 v + \theta^3}{v(v+2)(v+4)} \]
\[ + 2 \left( \frac{1}{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})} \right)^3 \]
\[ + 6 \left( \frac{1}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})} \right) \frac{(v+2)(v+4) + 3\theta v(v+2) + 3\theta^2 v + \theta^3}{v(v+2)(v+4)} \]
\[ + 6 \left( \frac{1}{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})} \right)^2 \]
\[ - 3 \left( \frac{1}{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})} \right) \]
\[ \beta_2 = \frac{\mu_4^3}{\mu_2^5} \]
\[ = \left( \frac{1}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})} \right) \frac{(v+2)(v+4) + 4\theta^2 v(v+2)(v+4)}{v(v+2)(v+4)} \]
\[ + 6 \theta^2 v(v+2) + 4\theta^3 v + \theta^4 \]
\[ + 4 \left( \frac{1}{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})} \right) \frac{(v+2)(v+4) + 3\theta v(v+2) + 3\theta^2 v + \theta^3}{v(v+2)(v+4)} \]
\[ - 3 \left( \frac{1}{\Gamma(1 + \frac{2}{\gamma})\Gamma(1 - \frac{2}{\gamma})} \right) \frac{(v+2)(v+4) + 3\theta v(v+2) + 3\theta^2 v + \theta^3}{v(v+2)(v+4)} \]
\[ + 2 \left( \frac{1}{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})} \right)^3 \]
\[ - 3 \left( \frac{1}{\Gamma(1 + \frac{1}{\gamma})\Gamma(1 - \frac{1}{\gamma})} \right) \]

\[ \text{Corollary 3.2.} \text{ If } v = 0 \text{ then all the values of } \mu_1', \sigma^2, \mu_2', \mu_3', \mu_4, \mu_3, \mu_4, \beta_1 \text{ and } \beta_2 \text{ are true for Loglogistic distribution with parameters } \alpha \text{ and } \gamma. \]

\[ \text{Remark 3.2.} \text{ The shape characteristics of the Chi-square mixture of loglogistic distribution are determined by} \]
\[ \text{i. coefficient of skewness} \]
\[ \gamma_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \]
\[ = \left( \Gamma \left( 1 + \frac{3}{\gamma} \right) \Gamma \left( 1 - \frac{3}{\gamma} \right) \right) \left[ \theta^3 + 3\theta^2 v + 3\theta v(v + 2) + v(v + 2)(v + 4) \right] \]
\[ - 3 \left[ (\theta + v) \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 - \frac{1}{\gamma} \right) \right] \]
\[ \left[ \Gamma \left( 1 + \frac{2}{\gamma} \right) \Gamma \left( 1 - \frac{2}{\gamma} \right) (\theta^2 + 2\theta v + v(v + 2)) \right] + 2 \left[ (\theta + v) \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 - \frac{1}{\gamma} \right) \right]^3 \]
\[ \left( \frac{1}{\Gamma \left( 1 + \frac{2}{\gamma} \right) \Gamma \left( 1 - \frac{2}{\gamma} \right) (\theta^2 + 2\theta v + v(v + 2)) - \left( (\theta + v) \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 - \frac{1}{\gamma} \right) \right)^2} \right) \]
\[ \gamma_2 = \beta_2 - 3 \]
\[ = \left( \Gamma \left( 1 + \frac{4}{\gamma} \right) \Gamma \left( 1 - \frac{4}{\gamma} \right) \right) \left[ \theta^4 + 4\theta^3 v + 6\theta^2 v(v + 2) + 4\theta v(v + 2)(v + 4) \right] \]
\[ + v(v + 2)(v + 4)(v + 6) \]
\[ - 4 \left[ (\theta + v) \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 - \frac{1}{\gamma} \right) \right] \]
\[ \left( \Gamma \left( 1 + \frac{3}{\gamma} \right) \Gamma \left( 1 - \frac{3}{\gamma} \right) \right) \left[ \theta^3 + 3\theta^2 v + 3\theta v(v + 2) + v(v + 2)(v + 4) \right] \]
\[ + 6 \left[ \Gamma \left( 1 + \frac{2}{\gamma} \right) \Gamma \left( 1 - \frac{2}{\gamma} \right) (\theta^2 + 2\theta v + v(v + 2)) \right] \]
\[ \left[ (\theta + v) \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 - \frac{1}{\gamma} \right) \right]^2 - 3 \left[ (\theta + v) \Gamma \left( 1 + \frac{1}{\gamma} \right) \Gamma \left( 1 - \frac{1}{\gamma} \right) \right]^4 \]
\[ - 3 \]

Since the value of \( \gamma_1 \) and \( \gamma_2 \) are always positive, the density of Chi-square mixture of Loglogistic distribution is skewed to the right, hence the shape is leptokurtic.

Finally, we will give the next definition whose results will generalized the previous results of Chi-square mixture distributions of Pareto and Loglogistic.

**Definition 3.3.** A random variable \( X \) is said to have a Chi-square mixture of Burr distribution if the density function is given by
\[
f_X(x; v, \alpha, \theta, \gamma) = \int_0^\infty e^{-\frac{x^2}{2}(\chi^2)} \frac{\alpha \gamma (\frac{x}{\theta + x})^\gamma}{2^\frac{\gamma}{2} \Gamma \left( \frac{\gamma}{2} \right)} \frac{\alpha \gamma (\frac{x}{\theta + x})^\gamma}{x[1 + (\frac{x}{\theta + x})^\gamma]^{\alpha/\gamma + 1}} d\chi^2, \quad 0 < x < \infty,
\]
with \( v \) degrees of freedom and parameters \( \alpha, \gamma, \) and \( \theta \) such that \( \int_{-\infty}^\infty f_X(x; v, \alpha, \theta, \gamma) = 1. \)

**Theorem 3.3.** If \( X \) follows a Chi-square mixture of Burr distribution with \( v \) degrees of freedom and parameters \( \alpha, \gamma, \) and \( \theta, \) then the \( r \)th raw moment about the
Moreover, the Kurtosis, \(\beta_2\) and the variance is
\[
\mu_\nu = \frac{\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{2\pi\Gamma(\frac{\nu}{2})\Gamma(\alpha)} \lim_{c \to \infty} \int_0^c e^{-\frac{x^2}{\nu}} (x^\nu)^{\frac{1}{2}-1} (\nu + \theta)^{-\frac{\nu}{2}} d\chi^2.
\]
Furthermore, the Skewness, \(\beta_1\), is
\[
\mu'_2 = \frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)} - \left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^2.
\]
and the variance is
\[
\sigma^2 = \frac{\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})(\theta^2 + 2\theta v + v(v + 2))}{\Gamma(\alpha)} - \left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^2.
\]
Furthermore, the Skewness, \(\beta_1\), is
\[
\mu'_3 = \left(\frac{\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^3 \left(\frac{\theta^3 + 3\theta^2 v + 3\theta v(v + 2) + v(v + 2)(v + 4)}{\Gamma(\alpha)}\right)
- 3\left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^3 \left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^2
+ 2\left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^3
\]
Moreover, the Kurtosis, \(\beta_2\), is
\[
\mu'_4 = \left(\frac{\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^4 \left[\theta^4 + 4\theta^3 v + 6\theta^2 v(v + 2) + 4\theta(v + 2)(v + 4) + v(v + 2)(v + 4)(v + 6)\right]
- 4\left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^4 \left[\theta^4 + 4\theta^3 v + 6\theta^2 v(v + 2) + 4\theta(v + 2)(v + 4)\right]
+ 6\left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^4 \left[\theta^4 + 4\theta^3 v + 6\theta^2 v(v + 2)\right]
- 3\left(\frac{(\theta + v)\Gamma(1 + \frac{\nu}{2})\Gamma(\alpha - \frac{\nu}{2})}{\Gamma(\alpha)}\right)^4
\]
Proof. Let \(X\) be a random variable which has the density of Chi-square mixture of Burr distribution. Then the \(r\)th raw moment is given by
\[
\mu'_r = \int_0^\infty \int_0^\infty e^{-\frac{x^2}{\nu}} (\chi^\nu)^{\frac{1}{2}-1} (\nu + \theta)^{-\frac{\nu}{2}} d\chi^2.
\]
which is equivalent to,

\[ \mu'_r = \frac{\Gamma(1 + \frac{\alpha}{\gamma})\Gamma(\alpha - \frac{\alpha}{\gamma})}{2^\gamma \Gamma(\frac{\alpha}{\gamma})\Gamma(\alpha)} \lim_{b \to \infty} \int_0^b e^{-\frac{x^2}{2}} (x^2)^{\frac{\alpha}{\gamma} - 1} (x^2 + \theta)^r \, dx. \]

If \( r = 1 \), then Equation (11) becomes

\[ \mu'_1 = (\theta + v) \frac{\Gamma(1 + \frac{\alpha}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}. \]

Now, if \( r = 2 \), we have

\[ \mu'_2 = \frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)} \left[ v(v + 2) + 2\theta v + \theta^2 \right]. \]

Thus, the variance is given by

\[ \sigma^2 = \mu'_2 - \mu'_1^2 \]
\[ = \frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)} \left[ v(v + 2) + 2\theta v + \theta^2 \right] - \left[ (\theta + v) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)} \right]^2. \]

When \( r = 3 \), we obtain

\[ \mu'_3 = \frac{\Gamma(1 + \frac{3}{\gamma})\Gamma(\alpha - \frac{3}{\gamma})}{\Gamma(\alpha)} \left[ v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3 \right], \]

and when \( r = 4 \), we have

\[ \mu'_4 = \frac{\Gamma(1 + \frac{4}{\gamma})\Gamma(\alpha - \frac{4}{\gamma})}{\Gamma(\alpha)} \left[ v(v+2)(v+4)(v+6) + 4\theta v(v+2)(v+4) + 6\theta^2 v(v+2) + 4\theta^3 v + \theta^4 \right]. \]

Thus, using the values of \( \mu'_4, \mu'_3, \mu'_2 \) and \( \mu'_1 \) to obtained the third and fourth central moments, we then have

\[ \mu_3 = \left( \frac{\Gamma(1 + \frac{3}{\gamma})\Gamma(\alpha - \frac{3}{\gamma})}{\Gamma(\alpha)} \left[ v(v + 2)(v + 4) + 3\theta v(v + 2) + 3\theta^2 v + \theta^3 \right] \right) \]
\[- 3 \left( \theta + v \right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)} \left[ \frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)} (\theta^2 + 2\theta v + v(v + 2)) \right] \]
\[ + 2 \left( \theta + v \right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)} \left[ \frac{\Gamma(1 + \frac{3}{\gamma})\Gamma(\alpha - \frac{3}{\gamma})}{\Gamma(\alpha)} \left( v(v + 2) + 2\theta v + \theta^2 \right) \right]^3. \]
Hence, the skewness, $\beta_1$, is

$$\begin{align*}
\mu_4 &= \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \left[ v(v+2)(v+4)(v+6) + 4\theta v(v+2)(v+4)
+ 6\theta^2 v^2 (v+2) + 4\theta^3 v + \theta^4 \right]
- 4 \left( \theta + v \right) \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}
\left( \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \right)^2
- 3 \left( \theta + v \right) \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}
\left( \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \right)^3. \end{align*}$$

while the kurtosis, $\beta_2$, is

$$\begin{align*}
\mu_4 &= \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \left[ v(v+2)(v+4)(v+6) + 4\theta v(v+2)(v+4)
+ 6\theta^2 v^2 (v+2) + 4\theta^3 v + \theta^4 \right]
- 4 \left( \theta + v \right) \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}
\left( \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \right)^2
- 3 \left( \theta + v \right) \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}
\left( \frac{\Gamma(1 + \frac{1}{2})\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \right)^3.
\end{align*}$$

now the proof is done.
Corollary 3.3. If $v = 0$ then all the values of $\mu_1, \sigma^2, \mu_2, \mu_3, \mu_4, \mu_3, \mu_4, \beta_1$ and $\beta_2$ are true for Burr distribution with parameters $\alpha, \gamma$ and $\theta$.

Remark 3.3. The shape characteristics of the Chi-square mixture of Burr distribution are determined by

i. coefficient of skewness

$$\gamma_1 = \frac{\mu_3}{\sqrt{\mu_2}} = \left(\frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(\alpha - \frac{3}{\gamma})}{\Gamma(\alpha)}\right) \left[\theta^3 + 3\theta^2v + 3\theta v(v + 2) + v(v + 2)(v + 4)\right]$$

$$- 3 \left(\theta + v\right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}$$

$$\frac{\Gamma(1 + \frac{3}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)} \left(\theta^2 + 2\theta v + v(v + 2)\right) + 2 \left(\theta + v\right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}$$

$$\left(\frac{1}{\sqrt{\frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)} \left(\theta^2 + 2\theta v + v(v + 2)\right) - \left(\theta + v\right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}\right)^2}\right)$$

ii. coefficient of kurtosis

$$\gamma_2 = \beta_2 - 3 = \left(\frac{\Gamma(1 + \frac{3}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)}\right) \left[\theta^4 + 4\theta^3v + 6\theta^2 v(v + 2) + 4\theta v(v + 2)(v + 4)\right]$$

$$+ v(v + 2)(v + 4)(v + 6) - 4 \left(\theta + v\right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}$$

$$\frac{\Gamma(1 + \frac{3}{\gamma})\Gamma(\alpha - \frac{2}{\gamma})}{\Gamma(\alpha)} \left[\theta^3 + 3\theta^2v + 3\theta v(v + 2) + v(v + 2)(v + 4)\right]$$

$$+ 6 \frac{\Gamma(1 + \frac{2}{\gamma})\Gamma(\alpha - \frac{2}{gamma})}{\Gamma(\alpha)} \left(\theta^2 + 2\theta v + v(v + 2)\right)$$

$$\left(\theta + v\right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}\right)^2 - 3 \left(\theta + v\right) \frac{\Gamma(1 + \frac{1}{\gamma})\Gamma(\alpha - \frac{1}{\gamma})}{\Gamma(\alpha)}\right)^4 - 3.$$

Since the value of $\gamma_1$ and $\gamma_2$ are always positive, the density of Chi-square mixture of Burr distribution is skewed to the right and hence, the shape is leptokurtic.

Remark 3.4. Let $\gamma = 1$. Then Definition 3.3 becomes the Chi-square mixture of Pareto distribution for any given parameter $\alpha$. Similarly, if $\alpha = 1$, it becomes the Chi-square mixture of Loglogistic distribution. Consequently, their results follows from the Chi-square mixture of Burr distribution. Hence, Chi-square mixture of Burr distribution is a generalization of the two mixed distribution.
References


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